# Variation of the power coupled to the mode groups of a circular core square law multimode fibre from a circular single spatial mode laser

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This analysis uses standard theory for graded-index multimode fibre having an axial-symmetric refractive index profile [1-6] to investigate the power coupled from a circular, single spatial mode, monochromatic laser into the multimode fibre. It is shown that:

- The excited mode power distribution (MPD) is independent of the orientation of the optical polarisation of the laser source.
- The coupled power is not equi-partitioned between the individual modes within a group.
- At the launch, the optical polarisation of the excited fibre modes are the same as that of the exciting laser.
- For axial-symmetric refractive index distributions the impulse response remains constant when the angle of optical polarization is rotated.

## Laser model

In this document the laser is modelled as a Gaussian, single, spatial mode beam. Therefore, the electric field of the laser, E(x,y), is given by:

$$E(x,y) = E(x)\cdot E(y) \tag{1}$$

where:

$$E(x) = \frac{1}{\sqrt{\omega 0}}\cdot\left(\frac{2}{\pi}\right)^{\frac{1}{4}}\cdot\exp\left[-\left(\frac{x-\delta}{\omega 0}\right)^2\right] \tag{2}$$

and

$$E(y) = \frac{1}{\sqrt{\omega 0}}\cdot\left(\frac{2}{\pi}\right)^{\frac{1}{4}}\cdot\exp\left[-\left(\frac{y-\varepsilon}{\omega 0}\right)^2\right] \tag{3}$$

$\omega 0$ is the $1/e^2$ waist of the laser beam, $\delta$ and $\varepsilon$ are the x and y offsets of the laser.

## Fibre modes

Initially, for mathematical convenience, Hermite-Gaussian (HMG) functions are used as basis functions for the fibre modes. These modes are useful in the following ways:

- For excitation with optically polarised sources HMG modes simplify the deduction of the initial state of polarisation of the excited modes[1].
- They lead to simple analytical equations for the power coupling coefficients as a function of source polarisation.

Later, the HMG modes and associated power coupling coefficients are transformed into the more usual Laguerre-Gaussian modes[2,3] and associated coupling coefficients:

### HMG fibre modes

The HMG modes of the fibre, $\Psi(x,y,p,q,\Omega 0)$, are given by:

$$\Psi(x,y,p,q,\Omega 0) = \psi(x,p,\Omega 0)\cdot\psi(y,q,\Omega 0) \tag{4}$$

where:

$$\psi(x,p,\Omega 0) = \left(\frac{2}{\pi}\right)^{\frac{1}{4}} \cdot \left(\frac{1}{\sqrt{2^p \cdot p! \cdot \Omega 0}}\right) \cdot H\left(p, \sqrt{2} \cdot \frac{x}{\Omega 0}\right) \cdot e^{-\left(\frac{x}{\Omega 0}\right)^2} \qquad (5)$$

and

$$\psi(y,q,\Omega 0) = \left(\frac{2}{\pi}\right)^{\frac{1}{4}} \cdot \left(\frac{1}{\sqrt{2^q \cdot q! \cdot \Omega 0}}\right) \cdot H\left(q, \sqrt{2} \cdot \frac{y}{\Omega 0}\right) \cdot e^{-\left(\frac{y}{\Omega 0}\right)^2} \qquad (6)$$

where:

$\Omega 0$ is the $1/e^2$ waist of the fundamantel mode of the fibre,
p and q are the HMG mode indices,

and

$$H(p,x) = \sum_{m=0}^{\mathrm{floor}\left(\frac{p}{2}\right)} \frac{(-1)^m \cdot p! \cdot (2 \cdot x)^{(p-2 \cdot m)}}{m! \cdot (p-2 \cdot m)!} \qquad (7)$$

and

$$H(q,x) = \sum_{m=0}^{\mathrm{floor}\left(\frac{q}{2}\right)} \frac{(-1)^m \cdot q! \cdot (2 \cdot x)^{(q-2 \cdot m)}}{m! \cdot (q-2 \cdot m)!} \qquad (8)$$

## Field coupling coefficients

The mode coupling from the laser to an individual fibre mode, $C_{pq}$, can be calculated from the overlap integral between the laser field and the fibre modal field.

$$C_{pq} = \int_{-\infty}^{\infty} E(x) \cdot \psi(x,p,\Omega 0)\, dx \cdot \int_{-\infty}^{\infty} E(y) \cdot \psi(y,q,\Omega 0)\, dy \qquad (9)$$

or

$$C_{pq} = Cx_p \cdot Cy_q \qquad (10)$$

It can be shown that:

$$Cx_0 = \exp\left[\frac{-\left(\alpha 1^2 \cdot \alpha 2^2 \cdot \delta^2\right)}{2 \cdot \left(\alpha 1^2 + \alpha 2^2\right)}\right] \cdot \frac{\sqrt{2 \cdot \alpha 1 \cdot \alpha 2}}{\sqrt{\alpha 1^2 + \alpha 2^2}} \quad \text{and} \quad Cy_0 = \exp\left[\frac{-\left(\alpha 1^2 \cdot \alpha 2^2 \cdot \varepsilon^2\right)}{2 \cdot \left(\alpha 1^2 + \alpha 2^2\right)}\right] \cdot \frac{\sqrt{2 \cdot \alpha 1 \cdot \alpha 2}}{\sqrt{\alpha 1^2 + \alpha 2^2}} \qquad (11)$$

$$Cx_1 = \frac{\sqrt{2}}{A} \cdot (\alpha 2 \cdot \delta) \cdot Cx_0 \quad \text{and} \quad Cy_1 = \frac{\sqrt{2}}{A} \cdot (\alpha 2 \cdot \varepsilon) \cdot Cy_0 \qquad (12)$$

where:

$$\alpha 1 = \frac{\sqrt{2}}{\Omega 0} \qquad \text{and} \qquad \alpha 2 = \frac{\sqrt{2}}{\omega 0} \tag{13}$$

$$A = \frac{\alpha 2}{\alpha 1} + \frac{\alpha 1}{\alpha 2} \tag{14}$$

Although not detailed here, for higher order modes, similar analytical equations for the field coupling and power coupling coefficients can be derived.

## Mode group power (MGP)

The power coupled to an individual mode, $P_{pq}$, is given by:

$$P_{pq} = \left(Cx_p \cdot Cy_q\right)^2 \tag{15}$$

Let the mode group number or order be M:

$$M = p + q + 1 \tag{16}$$

The the power coupled to the mode group M denoted $MGP_M$ is:

$$MGP_M = \sum_{p=0}^{M-1} \left[Cx_p \cdot Cy_{(M-p-1)}\right]^2 \tag{17}$$

Since the fibre is circular we will initially assume there is no special direction that must be used to align the basis functions describing the fibre modes. Later we will see that for polarised sources there is a natural orientation. We introduce an angle, $\theta$, and calculate the coupling coefficients and the $MGP_M$ as a function of $\theta$.

Let the x axis of the basis functions be at an angle $\theta$ relative to a line passing through the optical centre of the fibre and the centre of the laser beam. Assume that the laser spot is offset by a distance R from the optical centre of the fibre. Then we can write:

$$R^2 = \delta^2 + \varepsilon^2 \tag{18}$$

and

$$\delta = R \cdot \cos(\theta) \tag{19}$$

and

$$\varepsilon = R \cdot \sin(\theta) \tag{20}$$

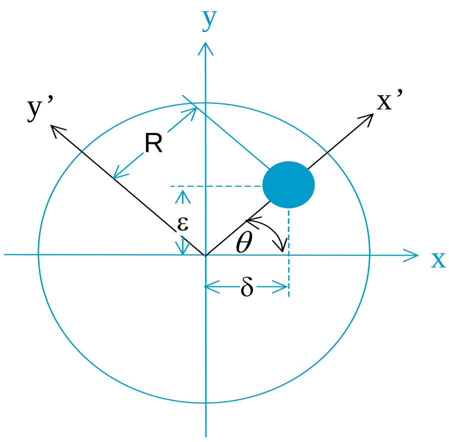



**Figure 1: Diagram of two basis mode axis**

## Calculation of $MGP_1$

Substituting for δ and ε from 19 and 20 and using equations 11 - 17 it can be shown that:

$$MGP_1 = \left(Cx_0 \cdot Cy_0\right)^2 = \exp\left(-\frac{\alpha 1^2 \cdot \alpha 2^2 \cdot R^2}{\alpha 1^2 + \alpha 2^2}\right) \cdot \left(\frac{2 \cdot \alpha 1 \cdot \alpha 2}{\alpha 1^2 + \alpha 2^2}\right)^2 \quad (21)$$

Therefore, $MGP_1$ is independent of the orientation of the axes of the basis modes.

## Calculation of $MGP_2$

Substituting for δ and ε from 19 and 20 and using equations 11 - 17 it can be shown that:

$$MGP_2 = \left(Cx_0 \cdot Cy_1\right)^2 + \left(Cx_1 \cdot Cy_0\right)^2 \quad (22)$$

$$\left(Cx_0 \cdot Cy_1\right)^2 = \exp\left(-\frac{\alpha 1^2 \cdot \alpha 2^2 \cdot R^2}{\alpha 1^2 + \alpha 2^2}\right) \cdot \left(\frac{2 \cdot \alpha 1 \cdot \alpha 2}{\alpha 1^2 + \alpha 2^2}\right)^2 \cdot \frac{2}{A^2} \cdot \alpha 2^2 \cdot \varepsilon^2 \quad (23)$$

$$\left(Cx_1 \cdot Cy_0\right)^2 = \exp\left(-\frac{\alpha 1^2 \cdot \alpha 2^2 \cdot R^2}{\alpha 1^2 + \alpha 2^2}\right) \cdot \left(\frac{2 \cdot \alpha 1 \cdot \alpha 2}{\alpha 1^2 + \alpha 2^2}\right)^2 \cdot \frac{2}{A^2} \cdot \alpha 2^2 \cdot \delta^2 \quad (24)$$

Therefore:

$$MGP_2 = \exp\left(-\frac{\alpha 1^2 \cdot \alpha 2^2 \cdot R^2}{\alpha 1^2 + \alpha 2^2}\right) \cdot \left(\frac{2 \cdot \alpha 1 \cdot \alpha 2}{\alpha 1^2 + \alpha 2^2}\right)^2 \cdot \frac{2}{A^2} \cdot \alpha 2^2 \cdot R^2 \quad (25)$$

Therefore, $MGP_2$ is also independent of the orientation of the axes of the basis modes.
By similar means it can be shown that for all M, $MGP_M$ is independent of the orientation of the basis modes.

Equations 23 and 24 show that although the MGP is constant the power coupled into a particular mode within a group is dependent on the orientation of the chosen basis functions. Later, when we take polarisation into account, we will see the there is a preferred orientation for the basis functions.

## Mode power distribution

The mode power distribution (MPD) is described by the function MPD(M):

$$\mathrm{MPD}(M) = \mathrm{MGP}_M \tag{26}$$

Since $\mathrm{MGP}_M$ is independent of orientation of the basis modes the MPD is orientation independent too. But, remember the power launched into each group is generally not equi-partitioned amongst the modes of a group.

## Launch polarisation and a natural basis axis

When the optical polarisation of the source is considered there is a natural orientation for the basis functions. For a linearly polarised laser source one axis of the excited HMG basis functions must be parallel to the direction of the polarisation, see figure 2.

Also, the Laguerre-Gaussian (LGG) LGG modes of a mode group can be expressed in terms of the HMG modes of the same mode group[2,3]. Hence, by writing the LGG fields in terms of HMG modes we can deduce that initially the excited fibre modes have the same polarization as the exciting laser.

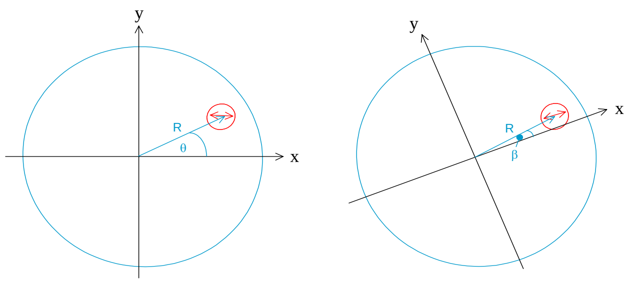



**Figure 2: Rotation of the direction of polarisation rotates the x and y axis of the HMG basis set.**

This means that if the position of the laser beam is kept fixed whilst the direction of optical polarisation is rotated then the HMG mode axes must also rotate, see figure 2. Therefore, rotation of the optical polarisation of the source is equivalent to changing the angle of the basis set in our previous calculations.

The transformation matrices from HMG to LGG modes for the first three mode groups are:

$$\mathrm{LGG}(0,0) = \mathrm{HMG}(0,0)$$

$$\begin{bmatrix} \mathrm{LGGc01} \\ \mathrm{LGGs01} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{HMG10} \\ \mathrm{HMG01} \end{bmatrix} \tag{27}$$

$$\begin{bmatrix} \mathrm{LGG10} \\ \mathrm{LGGc02} \\ \mathrm{LGGs02} \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{HMG20} \\ \mathrm{HMG02} \\ \mathrm{HMG11} \end{bmatrix}$$

These transformation equations have been used to calculate the distribution of launch power coupled to the LGG modes of the first three mode groups of a multimode fibre.

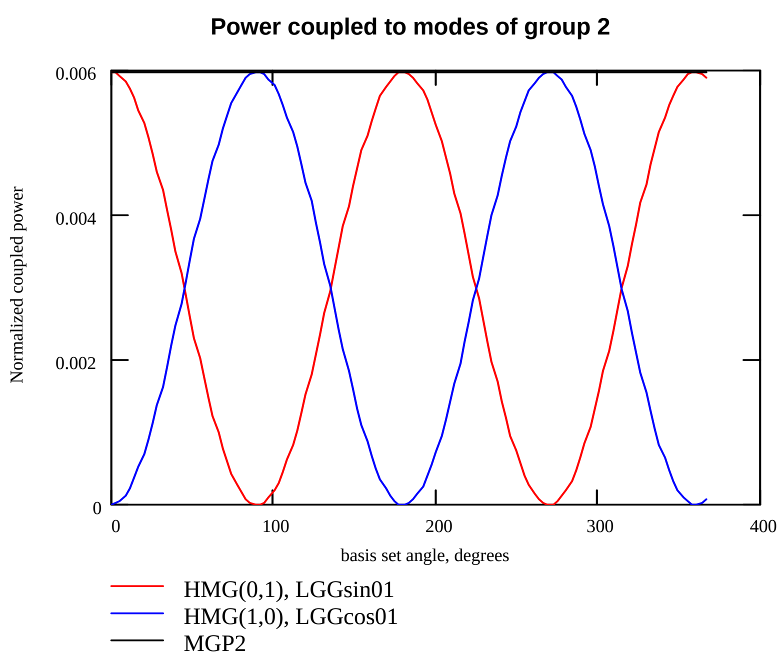



**Figure 3: Plots of the normalised power coupled to each mode of mode group 2 and $MGP_2$ for 15 um offset radius and $\omega 0$ of 4 um and $\Omega 0$ of 6.8 um (62 MMF).**

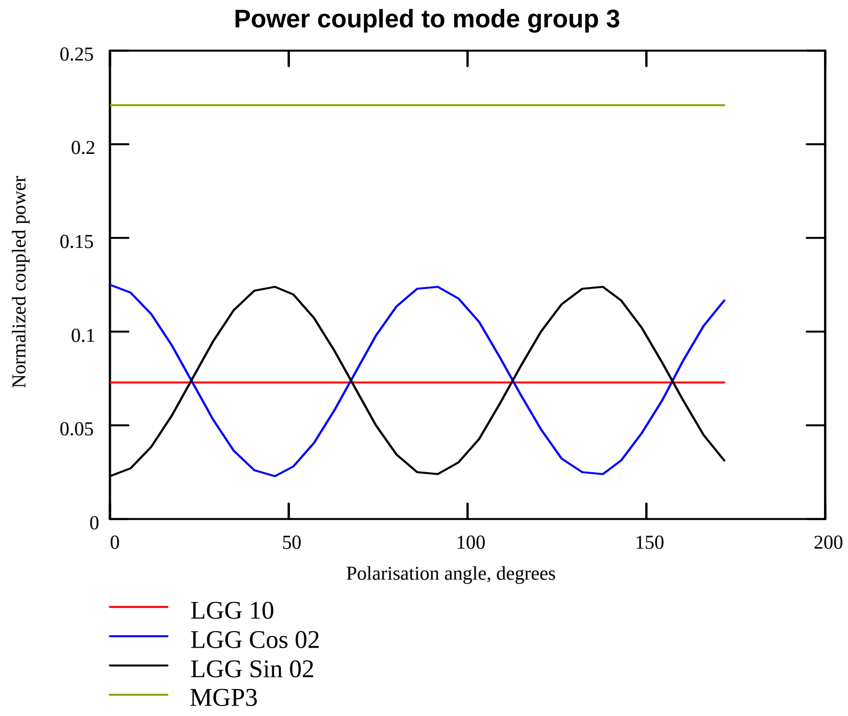



**Figure 4: Plots of the normalised power coupled to each mode of mode group 3 and $MGP_3$ for 10 um offset radius and $\omega$0 of 4 um and $\Omega$0 of 6.8 um (62 MMF).**

As a function of the basis set angle (or for a fixed spot the polarisation angle) figures 3 and 4 show the individual power couplings for the modes within groups 2 and 3 and the sum of the power of the modes of each group (MGP2 and MGP3). It is clearly seen that whilst the MGP is constant the power is not generally equi-partitioned at the launch point.

## Variation in impulse response as a function of launch polarisation.

For cylindrical symmetric refractive index profiles, the modes within a mode group that exchange power with each other as the polarization angle is changed have the same propagation constants and delay times. Therefore, according to the theory presented in this document for cylindrical symmetric refractive index profiles the impulse response is independent of launch polarisation.

Therefore, the experimentally observed variation in impulse response must be caused by some other effect. The variation in impulse response is not predicted by standard multimode fibre launch theory for cylindrical symmetric refractive index profiles.

## Conclusion

When a circular core, multimode fibre, with a square law refractive index variation is excited by a circular laser beam it has been shown that if standard wave equation theory is used that:

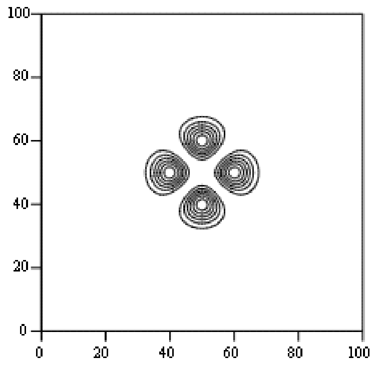
- The mode power distribution (MPD) is independent of polarisation of the laser launch.
- The launched power is not initially equi-partitioned between the individual modes within a group.
- Rotation of the launch polarisation vector causes the basis set axis to rotate and hence changes the distribution of power between the modes of a mode group (but the total power coupled to the group remains constant).
- For axial-symmetric refractive index distributions (per the current 10GBASE-LRM models) the impulse response remains constant when the angle of optical polarization is rotated.

## References

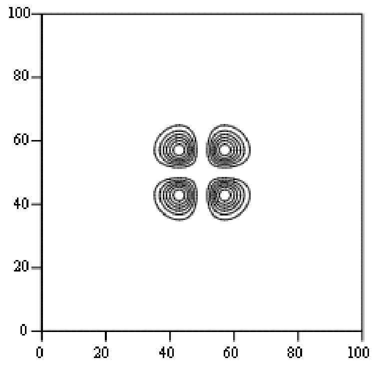
1) Arora R.K. and Zhong Lu, "Graphical depiction of the electromagnetic fields of Hermite-Gaussian modes", IEE Proceedings-H, Vol. 139, N0. 4. August 1992, pp 369-375.
2) Saijonmaa et al., "Selective excitation of parabolic-index optical fibers by Gaussian beams" Applied Optics Vol. 19 No.14, 15Jul1980, p.2442-2452.
3) Isidoro Kimel and Luis R. Elias, "Relations between Hermite and Laguerre Guassian modes", IEEE J-QE, Vol., 29, No 9, September 1993.
4) Grau et al., " Mode Excitation in Parabolic Index Fibres by Gaussian Beams" AEU, Band 34, 1980, Heft 6 pages 259-265.
5) Marcuse, D.: Light Transmission Optics, 2nd edition (Van Nostrand Reinhold, New York, 1982).
6) Ramo Simon, Whinnery John R. and Van Duzer Theodore: Fields and waves in communication electronics, 2nd edition (John Wiley & Sons, 1984), chapter 14.

## Other information

**Plots of the Intensity patterns for the three LGG modes of mode group 3**
**(x and y axes have arbitrary units (not um))**

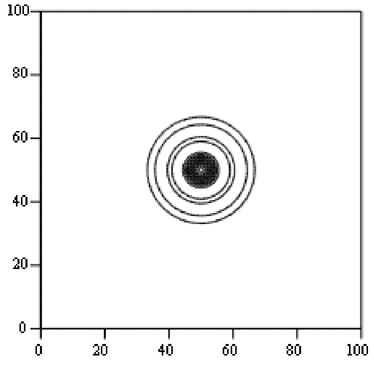



PLGC02




PLGS02

With cylindrically symmetric index perturbations these two modes have the same relative delay time.




PLG10

With cylindrically symmetric index perturbations this mode can have a different relative delay time when compared to the other two modes of the group.

**Equations for the first six HMG modes:**

$$\mathrm{HMG00}(x,y,w0) := \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\cdot\left(\frac{1}{w0}\right)\cdot\exp\left[-\left(\frac{x^2+y^2}{w0^2}\right)\right]$$

$$\mathrm{HMG10}(x,y,w0) := \frac{2\cdot\sqrt{2}}{\sqrt{\pi}}\cdot\frac{x}{w0^2}\cdot\exp\left[-\left(\frac{x^2+y^2}{w0^2}\right)\right]$$

$$\mathrm{HMG01}(x,y,w0) := \frac{2\cdot\sqrt{2}}{\sqrt{\pi}}\cdot\frac{y}{w0^2}\cdot\exp\left(-\frac{x^2+y^2}{w0^2}\right)$$

$$\mathrm{HMG11}(x,y,w0) := \frac{\sqrt{2}}{\sqrt{\pi}}\cdot\frac{1}{w0}\cdot\left(4\cdot\frac{x\cdot y}{w0^2}\right)\cdot\exp\left[\frac{-\left(x^2+y^2\right)}{w0^2}\right]$$

$$\mathrm{HMG20}(x,y,w0) := \frac{1}{\sqrt{\pi}}\cdot\frac{1}{w0}\cdot\left(4\cdot\frac{x^2}{w0^2}-1\right)\cdot\exp\left(-\frac{x^2+y^2}{w0^2}\right)$$

$$\mathrm{HMG02}(x,y,w0) := \frac{1}{\sqrt{\pi}}\cdot\frac{1}{w0}\cdot\left(4\cdot\frac{y^2}{w0^2}-1\right)\cdot\exp\left(-\frac{x^2+y^2}{w0^2}\right)$$

**Equations for the first six LGG modes:**

$$\mathrm{LG00}(x,y,w0) := \mathrm{HMG00}(x,y,w0)$$

$$\mathrm{LGS01}(x,y,w0) := \mathrm{HMG10}(x,y,w0)$$

$$\mathrm{LGC01}(x,y,w0) := \mathrm{HMG01}(x,y,w0)$$

$$\mathrm{LG10}(x,y,w0) := \frac{\sqrt{2}}{w0\cdot\sqrt{\pi}}\cdot\left[1-\frac{2\cdot\left(x^2+y^2\right)}{w0^2}\right]\cdot\exp\left[-\left(\frac{x^2+y^2}{w0^2}\right)\right]$$

$$\mathrm{LGC02}(x,y,w0) := \frac{2}{w0^3}\cdot\sqrt{\frac{2}{\pi}}\cdot\left(x^2-y^2\right)\cdot\exp\left[-\left(\frac{x^2+y^2}{w0^2}\right)\right]$$

$$\mathrm{LGS02}(x,y,w0) := \mathrm{HMG11}(x,y,w0)$$