Analytical Derivation: Noise Autocorrelation of a 4-Pole Bessel-Thomson Filter

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1 The Normalized Transfer Function

The 4th-order Bessel-Thomson transfer function, normalized for a DC gain of 1 (where $\omega_{3\text{dB,norm}} \approx 2.11392$ rad/s), is:

$$H(s) = \frac{105}{s^4 + 10s^3 + 45s^2 + 105s + 105} \tag{1}$$

2 Poles and Residue Evaluation

The poles p_i are the roots of the denominator of $H(s)$. Using the partial fraction expansion $H(s) = \sum_{i=1}^{4} \frac{A_i}{s-p_i}$, the residues A_i are calculated as $A_i = \text{Residue}[H(s), p_i] = \lim_{s\to p_i} (s - p_i)H(s)$.

i	p_i			A_i		
1	$-2.103\,79$	$+j$	$2.657\,42$	$-1.663\,39$	$+j$	$2.244\,08$
2	$-2.103\,79$	$-j$	$2.657\,42$	$-1.663\,39$	$-j$	$2.244\,08$
3	$-2.896\,21$	$+j$	$0.867\,234$	$1.663\,39$	$-j$	8.3963
4	$-2.896\,21$	$-j$	$0.867\,234$	$1.663\,39$	$+j$	8.3963

Table 1: Poles p_i and residues A_i.

3 Derivation of the Autocorrelation Function

The impulse response is defined as $h(t) = u(t)\sum_{i=1}^{4} A_i e^{p_i t}$, where $u(t)$ is the unit step function. The noise autocorrelation $R(\tau)$ is the convolution:

$$R(\tau) = \frac{N_0}{2}\int_{-\infty}^{\infty} h(t)h(t+\tau)dt = \frac{N_0}{2}\sum_{i=1}^{4}\sum_{j=1}^{4}\frac{A_iA_j}{-(p_i+p_j)}e^{p_j|\tau|} \tag{2}$$

4 Bandwidth Scaling and Sampling

For a filter with $f_{3\text{dB}} = 0.5R_s$ ($\omega_{3\text{dB}} = \pi R_s$), we must scale frequency by the scaling factor $\omega_{3\text{dB}}/\omega_{3\text{dB,norm}} \approx \pi R_s/2.11392$. This maps the sampling time $t_k = k/R_s$ to the normalized time $\tau_k = \pi k/2.11392 \approx 1.48615k$. Substituting this into (2):

$$r_k = \frac{N_0}{2} \sum_{i=1}^{4} \sum_{j=1}^{4} \frac{A_i A_j}{-(p_i + p_j)} e^{p_j(1.48615k)} \tag{3}$$

5 Normalized Autocorrelation Sequence

Normalizing by r_0 (where $\rho_k = r_k/r_0$), we obtain the sequence:

Lag (k)	Normalized Autocorrelation (ρ_k)
0	1.0000
1	0.0206
2	0.0013
3	-0.0001
4	1.7288e-6